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AN EXAMPLE CONCERNING FIXED POINTS

BY

A. GENEL AND J. LINDENSTRAUSS

ABSTRACT

An example is given of a contraction T defined on a bounded closed convex subset of Hilbert space for which $((I+T)/2)^n$ does not converge.

It is a well known fact that if K is a closed bounded and convex set in a uniformly convex Banach space and if T: $K \to K$ satisfies $\| Tx - Ty \| \leqq \| x - y \|$ for all x, $y \in K$, then T has a fixed point (cf. [1] or [3]). We are concerned here with an iteration method which was proposed for finding a fixed point for such a T. Krasnoselski [5] observed that if $x_1 \in K$ is arbitrary and if T has a compact range, then the sequence $\{x_n\}_{n=1}^{\infty}$ defined inductively by

$$x_n = (x_{n-1} + Tx_{n-1})/2, \quad n = 2, 3, \cdots \tag{1}$$

converges in the norm topology to a point which necessarily is a fixed point of T. Opial [6] showed that for an arbitrary contraction T in Hilbert space and, more generally, in a suitable class of uniformly convex spaces, the sequence given by (1) converges weakly to a fixed point of T. In [2] Kaniel proposed an iteration method which is more complicated than (1), which converges in the norm topology for an arbitrary contraction T. All these results suggest naturally the question whether the sequence given in (1) also converges always in the norm topology (of course under the assumptions made above on K and T). As mentioned in [2], the second named author constructed several years ago a quite artificial example which shows that the answer is negative. Since several mathematicians were interested in the details of this example and since presumably no "natural" counter-example to the convergence of (1) is known, it was decided to write up the example. The present exposition of the example was done by the first named author as part of his Master's thesis at the Hebrew University.

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EXAMPLE. There is a closed bounded and convex set K in the Hilbert space l_2, a contraction T of K into itself and a point $x_1 \in K$ such that the sequence $\{x_n\}_{n=1}^{\infty}$ defined by (1) does not converge in the norm topology.

PROOF. We shall define inductively a sequence $\{x_n\}_{n=1}^{\infty}$ in l_2 and a map T on this sequence so that (1) holds, the sequence $\{x_n\}_{n=1}^{\infty}$ is bounded but does not converge, and $\| Tx_i - Tx_j \| \leqq \| x_i - x_j \|$ for every pair of integers i, j. The points will be defined in the following order: x_1, Tx_1, x_2 (by (1)), Tx_2, x_3 (by 1)) and so on. Once this is done the desired example is obtained simply by employing Kirzbraun's theorem [4]. This theorem ensures that the map T which is defined only on $\{x_n\}_{n=1}^{\infty}$ can be extended to a map (still denoted by T) from l_2 into $K = \overline{\text{con}}\{x_n \cup Tx_n\}_{n=1}^{\infty}$ so that $\| Tx - Ty \| \leqq \| x - y \|$ for all $x, y \in l_2$.

Let $\{e_k\}_{k=1}^{\infty}$ be an orthonormal basis of l_2. We start the construction of the sequence $\{x_n\}$ by picking $x_1 = e_1$. The points $\{x_i\}_{i=1}^{n_1}$, with a suitable integer n_1, will be chosen in the plane P_1 determined by O, e_1 and e_2 according to the following rules:

$$\| x_i \| = \| Tx_i \|, \quad i = 1, 2, \cdots, n_1 - 1, \tag{2}$$

$$(x_i, Tx_i)/\| x_i \|^2 = \cos 2\varphi_1, \quad i = 1, 2, \cdots, n_1 - 1, \tag{3}$$

where φ_1 is an angle independent of i. Requirement (3) means that the angle between x_i and Tx_i (or more precisely between the ray Ox_i and OTx_i) is independent of i (see Fig. 1).

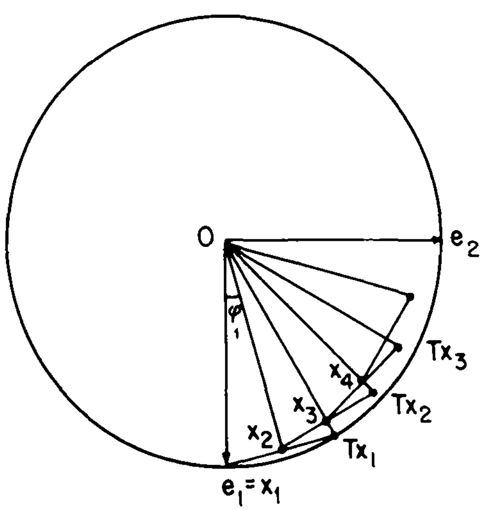

Fig. 1

It is clear from (2) and (3) (see Fig. 1) that for every $1 \leqq i,\ j \leqq n_1 - 1$ the triangle $(O,\ x_i,\ x_j)$ is congruent to the triangle $(O,\ Tx_i,\ Tx_j)$ and hence

(4) $$\| x_i - x_j \| = \| Tx_i - Tx_j \|, \quad 1 \leqq i, j \leqq n_1 - 1.$$

We have not yet chosen the angle φ_1 and the integer n_1. We pick them so that

(5) $$\varphi_1 = \pi/3(n_1 - 1),\ n_1 > 10, (\cos \varphi_1)^{n_1} \geqq 3/4.$$

This is possible since $\lim_{k \to \infty}(\cos \pi/3k)^k = 1$.

We define next the point Tx_{n_1}. Let y_1 be the point in the plane P_1 so that $\| y_1 \| = \| x_{n_1} \|$ and so that the angle between x_{n_1} and y_1 is $2\varphi_1$. Let $z_1 = (y_1 + x_{n_1})/2$. It is clear (see Fig. 2) that $\| z_1 - Tx_i \| < \| x_{n_1} - x_i \|$ for $1 \leqq i \leqq n_i - 1$. Hence there is a small positive λ_1 so that if we define $Tx_{n_1} = z_1 + \lambda_1 e_3$ we get that

(6) $$\| Tx_{n_1} - Tx_i \| < \| x_{n_1} - x_i \|, \quad 1 \leqq i < n_1.$$

Since $\| z_1 \| < \| x_{n_1} \|$, we can ensure also that

(7) $$\| Tx_{n_1} \| < \| x_{n_1} \|.$$

It is clear from the construction and from (5) (see also Fig. 2) that

(8) $$\| x_{n_1+1} \| = \| (x_{n_1} + Tx_{n_1})/2 \| \geqq \| z_1 \| \geqq 3/4$$

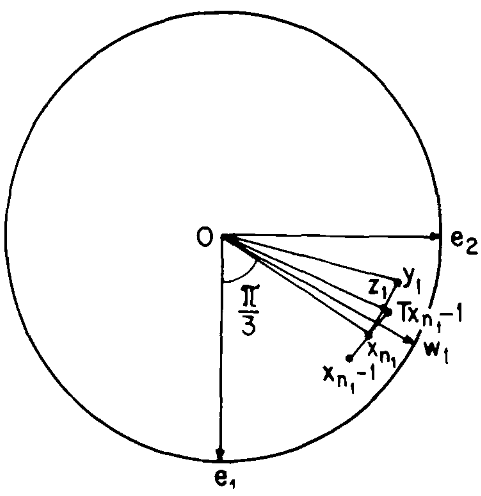

Fig. 2

and that the angle between x_1 and w_1 (= the unit vector in the direction of $(x_{n_1} + z_1)/2$) is between $\pi/3$ and $2\pi/5$. Also we have that

(9) $$\|\alpha w_1 - Tx_i\| < \|\alpha w_1 - x_i\| \text{ if } \alpha > 0 \text{ and } 1 \leqq i \leqq n_1 - 1.$$

Let P_2 be the plane determined by O, w_1 and e_3. The point x_{n_1+1} belongs to P_2 and so will all the points $\{x_i\}_{i=n_1+2}^{n_2}$ which we construct next.

Let $\bar{x}_{n_1}$ and $\bar{T}x_{n_1}$ be the orthogonal projections of x_{n_1}, respectively Tx_{n_1}, on the plane P_2. It is clear that

(10) $$\|x_{n_1} - \bar{x}_{n_1}\| = \|Tx_{n_1} - \bar{T}x_{n_1}\|.$$

In view of (9) and of the fact that every point in R_2 (see Fig. 3) is of the form $\alpha w_1 + \beta e_3$ with $\alpha \geqq \frac{1}{2}\cos(2\pi/5)$, we have that

(11) $$\varepsilon_1 = \min_{\substack{u \in R_2 \\ 1 \leqq i \leqq n_1 - 1}} (\|u - x_i\| - \|u - Tx_i\|) > 0.$$

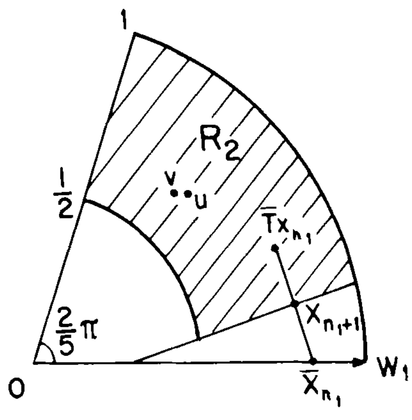

Fig. 3

We repeat now the procedure used for constructing $\{x_i\}_{i=1}^{n_1}$ by starting with x_{n_1+1} and rotating always in the plane P_2 by a fixed angle $2\varphi_2$. More precisely, we take for $n_1 < i < n_2$

(12) $$\|Tx_i\| = \|x_i\|, \quad (x_i, Tx_i) = \|Tx_i\|^2 \cos 2\varphi_2,$$

where φ_2 and n_2 are chosen so that

(13) $$\varphi_2 = \pi/3(n_2 - n_1 - 1),\ n_2 > n_1 + 10,\ \tfrac{3}{4}(\cos\varphi_2)^{n_2 - n_1} > 5/8,\ 4\sin\varphi_2 < \epsilon_1$$

and

(14) $$\|Tx_i - \bar{T}x_{n_1}\| < \|x_i - \bar{x}_{n_1}\|, \qquad n_1 < i < n_2.$$

That (13) is possible follows again from $\lim_{k\to\infty}(\cos \pi/3k)^k = 1$. That (14) is possible follows from a compactness argument: All the points x_i, $n_1 < i < n_2$, must be in R_2 (see Fig. 3) by (12) and (13). It is easily verified that for every $u \in R_2$ there is a $\delta(u) > 0$ so that if $v \in R_2$, $\| v \| = \| u \|$ and the angle between u and v is positive but not larger than $\delta(u)$, then $\| v - \bar{T}x_{n_1} \| < \| u - \bar{x}_{n_1} \|$.

Let us check that on its domain of definition till now T is a contraction, i.e. that $\| Tx_i - Tx_j \| \leqq \| x_i - x_j \|$ for $1 \leqq i, j < n_2$. For $1 \leqslant i, j \leqq n_1 - 1$ this follows from (4). For $1 \leqq i \leqq n_1 - 1$ and $j = n_1$ this follows from (6). For $n_1 < i, j < n_2$ we have by the same reasoning as the one which proved (4), that $\| Tx_i - Tx_j \| = \| x_i - x_j \|$. For $i = n_1$ and $n_1 < j < n_2$ the desired inequality follows from (10) and (14). Finally, let $1 \leqq i < n_1$ and $n_1 < j < n_2$. By (12) we get that $\| x_j - Tx_j \| < 2 \sin \varphi_2$ and hence, by (11) and (13),

$$\begin{aligned} \| Tx_i - Tx_j \| &\leqq \| Tx_i - x_j \| + \| x_j - Tx_j \| \\ &\leqq \| Tx_i - x_j \| + \varepsilon_1/2 < \| x_i - x_j \| \end{aligned}$$

as desired.

We continue now in an obvious inductive procedure. We define y_2, z_2 and w_2 in an obvious way and let $Tx_{n_2} = z_2 + \lambda_2 e_4$, with $\lambda_2 > 0$ but sufficiently small. Then $\| x_{n_2+1} \| \geqq 5/8$. All the points $\{x_i\}_{i=n_2+1}^{n_3}$ will be chosen to belong to the plane P_3 determined by O, w_2 and e_4. In order to show that the construction can be continued in exactly the same manner, we have just to observe that if R_3 is the domain in P_3 which is analogous to the domain R_2 in P_2 (see Fig. 3), then for $u \in R_3$ we have $u = \alpha w_1 + \beta e_3 + \gamma e_4$, where $a \geqq \frac{1}{2}(\cos(2\pi/5))^2$, and hence by (9)

$$\min_{\substack{u \in R_3 \\ 1 \leqq i \leqq n_1 - 1}} (\| u - x_i \| - \| u - Tx_i \|) > 0. \tag{15}$$

Also by the same reasoning which showed that (9) holds we get that

$$\| \alpha w_2 - Tx_i \| < \| \alpha w_2 - x_i \| \quad \text{if} \quad \alpha > 0 \quad \text{and} \quad n_1 < i < n_2 \tag{16}$$

and

$$\| \alpha w_2 - \bar{T}x_{n_1} \| < \| \alpha w_2 - \bar{x}_{n_1} \| \text{ if } \alpha > 0. \tag{17}$$

From (10), (15), (16) and (17) we deduce that

$$\epsilon_2 = \lim_{\substack{u \in R_3 \\ 1 \leqq i \leqq n_2 - 1}} (\| u - x_i \| - \| u - Tx_i \|) > 0,$$

and now it is clear how to continue the inductive definition of $\{x_i\}_{i=1}^{\infty}$.

The sequence $\{x_i\}_{i=1}^{\infty}$ is bounded in norm (by 1) and also bounded from below in norm (by 1/2). The sequence does not converge in norm, however, since, as easily seen, $x_i \to 0$ weakly.

Let us observe that by Opial's result which was mentioned in the introduction every extension of T from $\{x_i\}_{i=1}^{\infty}$ to a contraction on l_2 must leave the origin fixed.

The referee brought to our attention the following additional bibliographical information. The fixed point theorem mentioned in the first sentence of the paper was also proved independently by D. Göhde, Math. Nachr. **30** (1965), 251–258. The result of Krasnoselski is true also in the more general case in which it is merely assumed that $I - T$ is a closed map (besides of course T being a contraction in a uniformly convex space). This was observed by F. E. Browder and W. V. Petryshyn, Bull. Amer. Math. Soc. **72** (1966), 571–575.

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INSTITUTE OF MATHEMATICS
THE HEBREW UNIVERSITY OF JERUSALEM
JERUSALEM, ISRAEL